

DEFORMATION OF F -INJECTIVITY AND LOCAL COHOMOLOGY

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ABSTRACT. We give sufficient conditions for F -injectivity to deform. We show these conditions are met in two geometrically interesting settings, namely when the special fiber has isolated non-CM locus or is F -split.

1. INTRODUCTION

A central and interesting question in the study of singularities is how they behave under deformation. Given a local ring of positive characteristic, view this ring as the total space of a fibration. The special fiber of this fibration is a hypersurface in R , i.e., a variety with coordinate ring R/xR where $x \in R$ is a regular element. An important question is if the singularity type of the total space R is no worse than the singularity type as the special fiber. This deformation question has been studied in detail for singularities defined by Frobenius [Fed83, Sin99b] where it is noted that F -rationality deforms always and both F -purity and F -regularity fail to deform in general. An important and outstanding conjecture is that F -injectivity deforms in general. Deformation in the Cohen-Macaulay case is known [Fed83]. The general conjecture is supported by recent work showing the characteristic 0 analogue of this singularity type, called *Du Bois singularities*, deform [KS11]. Recall that a local ring (R, \mathfrak{m}) of prime characteristic $p > 0$ is *F -injective*, if the Frobenius action on the local cohomology $H_{\mathfrak{m}}^i(R)$, induced by the Frobenius map on R , is injective for all $i \geq 0$.

Main Theorem. (*Theorem 3.6*) *Let (R, \mathfrak{m}, k) be a local ring of prime characteristic $p > 0$ and $x \in \mathfrak{m}$ a regular element. If R/xR is F -injective and for each $\ell > 0$ and $i \geq 0$ the homomorphism $H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/xR)$ induced by the natural surjection $R/x^\ell R \rightarrow R/xR$, is surjective, then R is F -injective.*

We show in particular that this hypothesis is satisfied when the length of the local cohomology modules $H_{\mathfrak{m}}^i(R/xR)$ is finite for $i < \dim R - 1$; a condition called *finite length cohomology*. Geometrically, it is the condition that the non Cohen-Macaulay locus on the special fiber is isolated and this combination shows that F -injectivity deforms under mild geometric criteria in low dimensions, see Corollary 4.7.

Main Theorem. (*Corollary 4.6*) *Let (R, \mathfrak{m}, k) be a local ring of characteristic $p > 0$ with perfect residue field and $x \in \mathfrak{m}$ a regular element. If R/xR has FLC and is F -injective, then R is F -injective.*

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Also, utilizing the recent work of L. Ma on a condition known as *anti-nilpotence*, we demonstrate the following deformation theoretic relationship between F -injectivity and F -splitting, which is equivalent to F -purity under the F -finiteness hypothesis on the local ring.

Main Theorem. (*Theorem 4.11*) *Let (R, \mathfrak{m}, k) be a local ring of characteristic $p > 0$ and $x \in \mathfrak{m}$ a regular element. If R/xR is F -split, then R is F -injective.*

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Conventions: Unless otherwise stated all rings are noetherian and of characteristic $p > 0$ where p is a prime integer.

2. PRELIMINARIES AND NOTATIONS

2.1. Notation. For a ring R of characteristic $p > 0$, the Frobenius is the map $F: R \rightarrow R$ sending an element to its p -th power. For an R -module M , denote by $F_*M = \{F_*m: m \in M\}$, called the *Frobenius pushforward* of M , namely $M = F_*M$ as underlying abelian groups, but with its R -module structure twisted by Frobenius: If $r \in R$ and $F_*m \in F_*M$, then $r \cdot F_*m = F_*(r^p m)$. We also denote the e -th iterate of the Frobenius pushforward of M by $F_*^e(M)$. The functor F_*^e is exact and commutes with localization.

2.2. Local cohomology. For a more complete introduction see [ILL]. Fix a ring R and an ideal I , and let M be an R -module; not necessarily noetherian. The local cohomology module supported at I is $H_I^i(M) = \varinjlim_t \text{Ext}_R^i(R/I^t, M)$. When I is generated up to radical by g_1, \dots, g_n , one may compute $H_I^i(M)$ as the i -th cohomology of the Čech complex with respect to I , denoted $\check{C}^\bullet(M; I)$:

$$0 \rightarrow M \rightarrow \oplus_i M_{g_i} \rightarrow \oplus_{i < j} M_{g_i g_j} \rightarrow \cdots \rightarrow M_{g_1 \cdots g_n} \rightarrow 0.$$

We briefly discuss iterated local cohomology as it plays a role in the proof of Theorem 3.6. For more detail see [Har67]. Given two ideals I and J in R , and an R -module M , let $\check{C}^\bullet(M; I)$ (resp. $\check{C}^\bullet(M; J)$) be the Čech complex of M with respect to I (resp. with respect to J). Considering $\check{C}^\bullet(M; I)$ as the horizontal complex and $\check{C}^\bullet(M; J)$ as the vertical complex, one obtains a double complex $C^{\bullet\bullet} = \check{C}^\bullet(M; I) \otimes_R \check{C}^\bullet(M; J)$. This double complex is the first page of a spectral sequence $E_0^{p,q}$, called the *local cohomology spectral sequence*. For more on spectral sequences see [Wei94]. The convergence of this spectral sequence is known.

Theorem 2.1. (*Convergence of local cohomology spectral sequence* [Har67, Prop. 1.4]) *For I and J ideals in a ring R and M an R -module,*

$$E_2^{p,q} = H_J^p(H_I^q(M)) \Rightarrow E_\infty^{p,q} = H_{I+J}^{p+q}(M).$$

Using this theorem, it is easy to compute an isomorphism that we need.

Lemma 2.2. *For (R, \mathfrak{m}, k) a local ring with $x \in \mathfrak{m}$ a regular element, then for all $i \geq 0$, $H_{\mathfrak{m}}^i(H_{(x)}^1(R)) \cong H_{\mathfrak{m}}^{i+1}(R)$.*

Proof. First, note that $H_{(x)}^q(R)$ is nonzero only when $q = 1$. Thus the $E_2^{p,q}$ page of the spectral sequence computing the double complex $H_{\mathfrak{m}}^\bullet(H_{(x)}^\bullet(-))$ degenerates. By Theorem 2.1, $E_2^{p,q} = H_{\mathfrak{m}}^p(H_{(x)}^q(R))$ and $E_\infty^{p,q} = H_{\mathfrak{m}}^{p+q}(R)$ for all $p \geq 0$ and $q \geq 0$. Since the sequence degenerates at the $E_2^{p,q}$ page, we have $H_{\mathfrak{m}}^p(H_{(x)}^q(R)) = E_2^{p,q} = E_\infty^{p,q} = H_{\mathfrak{m}}^{p+q}(R)$ for all $p \geq 0$ and $q \geq 0$. Applying this with $p = i$ and $q = 1$ gives the result. \square

It is often easier to study spectral sequences as composition of derived functors; see [Lip02] for explicit details about derived categories and local cohomology. We summarize what we need. For an abelian category \mathcal{A} , denote by $K(\mathcal{A})$ the category of complexes in \mathcal{A} up to homotopic equivalence and $\mathbf{D}(\mathcal{A})$ its derived category. For R a ring, denote by $R\text{-mod}$ the category of R -modules. Let $I \subseteq R$ an ideal and $\mathcal{A} = R\text{-mod}$. One realizes the i -th local cohomology module with support in I as a functor $H_I^i: K(R\text{-mod}) \rightarrow R\text{-mod}$ which takes quasi-isomorphisms in $K(R\text{-mod})$ to isomorphisms in $R\text{-mod}$ and so it can be regarded as a functor on $\mathbf{D}(R\text{-mod})$. Denote by Γ_I the I -torsion functor. The right derived functor $\mathbf{R}\Gamma_I: \mathbf{D}(R\text{-mod}) \rightarrow \mathbf{D}(R\text{-mod})$ has the information of taking all of the local cohomology modules H_I^i at once and each H_I^i can be recovered in a functorial way from $\mathbf{D}(R\text{-mod})$ by taking the i -th cohomology of the image of $\mathbf{R}\Gamma_I$. The spectral sequence in Theorem 2.1 can be understood as a consequence of the Grothendieck spectral sequence theorem [Wei94, Cor. 10.8.3] stating that $\mathbf{R}\Gamma_I \circ \mathbf{R}\Gamma_J \cong \mathbf{R}\Gamma_{I+J}$. This equivalence will be utilized in Theorem 3.6

2.3. Frobenius linear maps. A central topic in this article is that of Frobenius linear maps. These are thoroughly explored in [HS77] under the name p -linear maps. We review the topic.

Definition 2.3. Let R be a commutative ring of characteristic p . For R -modules M and N , a *Frobenius linear map* is an element of $\text{Hom}_R(M, F_*N)$. More specifically, it is an additive map $\rho: M \rightarrow F_*M$ such that $\rho(ra) = r^p\rho(a)$ for any $r \in R$ and $a \in M$. If $M = N$, we call $\rho: M \rightarrow F_*M$ a *Frobenius action* on M .

Since F_* commutes with localization, given a Frobenius linear map between M and N there is an induced Frobenius linear map $H_{\mathfrak{m}}^i(M) \rightarrow F_*H_{\mathfrak{m}}^i(N)$ for each $i \geq 0$. One can make this explicit utilizing Čech resolutions as in Example 2.4. In categorical terms, a Frobenius linear map $\rho: M \rightarrow F_*N$ induces a morphism $\mathbf{R}\Gamma_I(\rho): \mathbf{R}\Gamma_I(M) \rightarrow \mathbf{R}\Gamma_I(F_*N) \cong F_*\mathbf{R}\Gamma_I(N)$ where $I \subseteq R$ is an ideal and the last isomorphism follows as F_* is exact. In particular, the Frobenius map on R thought of as a Frobenius action $\rho_F: R \rightarrow F_*R$ induces a natural Frobenius action on the local cohomology

$$\mathbf{R}\Gamma_I(\rho_R): \mathbf{R}\Gamma_I(R) \rightarrow F_*\mathbf{R}\Gamma_I(R).$$

This Frobenius action can be computed explicitly using Čech complexes.

Example 2.4. Consider (R, \mathfrak{m}, k) a local ring with $x \in \mathfrak{m}$ a regular element. Each term of the Čech complex

$$0 \rightarrow R \rightarrow R_x \rightarrow 0$$

has a Frobenius linear maps induced from the Frobenius on R . Therefore we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & R_x & \longrightarrow & 0 \\ \downarrow & & \downarrow \rho_F & & \downarrow \rho_F & & \downarrow \\ 0 & \longrightarrow & F_*R & \longrightarrow & F_*(R_x) & \longrightarrow & 0 \end{array}$$

Of course $H_{(x)}^0(R) = 0$ and $H_{(x)}^1(R) = R_x/R$. Taking cohomology we have the natural Frobenius linear map on $H_{(x)}^1(R) = R_x/R$. In particular, $\rho: H_{(x)}^i(R) \rightarrow F_*H_{(x)}^i(R)$ is the natural Frobenius $R_x/R \rightarrow F_*(R_x/R)$.

We see immediately the benefit of studying Frobenius linear maps on finite length modules when the residue field is perfect.

Lemma 2.5. *Let (R, \mathfrak{m}, k) be a local ring of prime characteristic $p > 0$ with perfect residue field and let M be an R -module of finite length, admitting an injective Frobenius action ρ . Then M is a finite dimensional k -vector space and ρ is a bijection.*

Proof. Since M has finite length, there exists $\ell > 0$ such that $\mathfrak{m}^\ell \cdot M = 0$. Fix $c \in \mathfrak{m}$. Then $\rho^e(c \cdot M) = c^{p^e} \cdot \rho(M) = 0$ for $p^e \geq \ell$. Since ρ is injective, $c \cdot M = 0$. Therefore, M is a finite dimensional k -vector space and ρ descends to an additive map on $M = M/\mathfrak{m}M$. Now since k is perfect and M is finite dimensional (M is of finite length) and ρ is injective, ρ must be bijective. \square

Remark 2.6. The perfectness of the residue field in Lemma 2.5 is necessary. In the case, $R = k$, the natural Frobenius action on the simple k -module k is bijective if and only if k is perfect. See also [Ene12, Cor. 7.7 and Prop. 7.12] in a similar vein.

3. PROOF OF THE MAIN THEOREM

We start with the following notation defining the key property about a regular element that we need to guarantee that F -injective deforms.

Definition 3.1. Let (R, \mathfrak{m}) be a local ring with $x \in \mathfrak{m}$ a regular element. Then we say that x is a *surjective element*, if the local cohomology map $H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/xR)$, which is induced by the natural surjection $R/x^\ell R \rightarrow R/xR$, is surjective for all $\ell > 0$ and $i \geq 0$.

Lemma 3.2. *Let (R, \mathfrak{m}) be a local ring of arbitrary characteristic. Assume that $x \in \mathfrak{m}$ is a surjective element. For each $\ell > 0$ and $j \geq \ell$, the multiplication map*

$$R/x^\ell R \xrightarrow{x^{j-\ell}} R/x^j R$$

induces an injection $H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/x^j R)$ for each $i \geq 0$.

Proof. Note that $R/x^\ell R \xrightarrow{x^{j-\ell}} R/x^j R$ is injective and it suffices to prove the lemma when $j = \ell + 1$ by an inductive argument. Consider the portion of the long exact sequence induced by $0 \rightarrow R/x^\ell R \xrightarrow{x} R/x^{\ell+1} R \rightarrow R/xR \rightarrow 0$:

$$H_{\mathfrak{m}}^{i-1}(R/x^{\ell+1} R) \xrightarrow{\beta_1} H_{\mathfrak{m}}^{i-1}(R/xR) \xrightarrow{\delta} H_{\mathfrak{m}}^i(R/x^\ell R) \xrightarrow{\beta_2} H_{\mathfrak{m}}^i(R/x^{\ell+1} R).$$

Since x is a surjective element, β_1 is surjective and hence δ is the zero map. This makes β_2 injective as desired. \square

Theorem 3.3. *Let (R, \mathfrak{m}, k) be a local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a surjective element. Assume that R/xR is F -injective and denote by*

$$\rho_{\ell,i}: H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow F_* H_{\mathfrak{m}}^i(R/x^{p^\ell} R),$$

the Frobenius linear map induced by the natural Frobenius $\rho_F: R/x^\ell R \rightarrow F_(R/x^{p^\ell} R)$. The map $\rho_{\ell,i}$ is injective for each $\ell > 0$ and $i \geq 0$.*

Proof. For every $\ell > 0$, the natural Frobenius map on $R/x^\ell R$ is a composition of ρ_F and a natural surjection π :

$$R/x^\ell R \xrightarrow{\rho_F} F_*(R/x^{p^\ell} R) \xrightarrow{\pi} F_*(R/x^\ell R).$$

Denote by $\rho_{\ell,i}: H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow F_* H_{\mathfrak{m}}^i(R/x^{p^\ell} R)$ the Frobenius linear map induced by ρ_F . We proceed by induction on ℓ to show that $\rho_{\ell,i}$ is injective for all $\ell > 0$. The case $\ell = 1$ is assured by hypothesis.

Assume $\ell > 1$ and consider the commutative diagram of R -modules with exact rows:

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R/x^{\ell-1}R & \xrightarrow{\cdot x} & R/x^\ell R & \longrightarrow & R/xR \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_*(R/x^{p(\ell-1)}R) & \xrightarrow{\cdot x^p} & F_*(R/x^{p\ell}R) & \longrightarrow & F_*(R/x^pR) \longrightarrow 0 \end{array}$$

where all vertical maps are the natural Frobenius linear maps. This induces the following commutative diagram of R -modules:

$$(3.2) \quad \begin{array}{ccccccc} H_{\mathfrak{m}}^{i-1}(R/xR) & \longrightarrow & H_{\mathfrak{m}}^i(R/x^{\ell-1}R) & \longrightarrow & H_{\mathfrak{m}}^i(R/x^\ell R) & \xrightarrow{\alpha} & H_{\mathfrak{m}}^i(R/xR) \\ \downarrow \rho_{1,i-1} & & \downarrow \rho_{\ell-1,i} & & \downarrow \rho_{\ell,i} & & \downarrow \rho_{1,i} \\ F_*H_{\mathfrak{m}}^{i-1}(R/x^pR) & \xrightarrow{F_*\delta_{i-1}} & F_*H_{\mathfrak{m}}^i(R/x^{p(\ell-1)}R) & \xrightarrow{F_*\beta} & F_*H_{\mathfrak{m}}^i(R/x^{p\ell}R) & \longrightarrow & F_*H_{\mathfrak{m}}^i(R/x^pR) \end{array}$$

The map $\alpha : H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/xR)$ is surjective, since x is a surjective element by assumption. From Lemma 3.2 and that F_* is exact, the map $F_*\beta$ is injective. Hence $F_*\delta_{i-1}$ is the zero map. Thus we have a commutative diagram

$$(3.3) \quad \begin{array}{ccccccc} H_{\mathfrak{m}}^i(R/x^{\ell-1}R) & \longrightarrow & H_{\mathfrak{m}}^i(R/x^\ell R) & \longrightarrow & H_{\mathfrak{m}}^i(R/xR) & \longrightarrow & 0 \\ \downarrow \rho_{\ell-1,i} & & \downarrow \rho_{\ell,i} & & \downarrow \rho_{1,i} & & \\ 0 & \longrightarrow & F_*H_{\mathfrak{m}}^i(R/x^{p(\ell-1)}R) & \longrightarrow & F_*H_{\mathfrak{m}}^i(R/x^{p\ell}R) & \longrightarrow & F_*H_{\mathfrak{m}}^i(R/x^pR) \end{array}$$

To complete the argument, apply the snake lemma to Diagram (3.3). This gives an exact sequence $\ker \rho_{\ell-1,i} \rightarrow \ker \rho_{\ell,i} \rightarrow \ker \rho_{1,i}$. Since $\rho_{1,i}$ is injective by F -injectivity of R/xR , and $\rho_{\ell-1,i}$ is injective by induction, we have that $\ker \rho_{\ell,i} = 0$. Hence $\rho_{\ell,i}$ is injective. \square

We record an easy lemma before the proof of the main theorem whose proof is left to the reader.

Lemma 3.4. *For a directed system $\{N_i, \tau_{i,j}\}_{i \in \Lambda}$ of R -modules, the system $\{F_*N_i, F_*\tau_{i,j}\}_{i \in \Lambda}$ is also directed and $F_*\varinjlim N_i \cong \varinjlim F_*N_i$.*

The next lemma explains the basic isomorphisms needed in the proof of the main theorem.

Lemma 3.5. *Let (R, \mathfrak{m}) be a local ring with $x \in \mathfrak{m}$ a regular element. Then for each $i > 0$, we have isomorphisms:*

$$H_{\mathfrak{m}}^i(H_{(x)}^1(R)) \cong H_{\mathfrak{m}}^{i+1}(R) \cong \varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^\ell R) \cong \varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{p\ell}R).$$

Proof. We show this by showing that all modules $H_{\mathfrak{m}}^{i+1}(R)$, $\varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^\ell R)$, and $\varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{p\ell}R)$ are isomorphic to the iterated local cohomology module $H_{\mathfrak{m}}^i(H_{(x)}^1(R))$. Computing $H_{(x)}^1(R)$ as

$$\varinjlim \{R/xR \xrightarrow{x} R/x^2R \xrightarrow{x} R/x^3R \xrightarrow{x} \dots\},$$

and noting that local cohomology commutes with direct limits, one has

$$\varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{\ell}R) \cong H_{\mathfrak{m}}^i(\varinjlim_{\ell} R/x^{\ell}R) \cong H_{\mathfrak{m}}^i(H_{(x)}^1(R)).$$

By Lemma 2.2,

$$\varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{\ell}R) \cong H_{\mathfrak{m}}^i(H_{(x)}^1(R)) \cong H_{\mathfrak{m}}^{i+1}(R).$$

Since $\{x^{p^{\ell}}\}_{\ell \in \mathbb{N}}$ is cofinal in $\{x^{\ell}\}_{\ell \in \mathbb{N}}$, one can compute $H_{(x)}^1(R)$ as the limit

$$\varinjlim \{R/x^p R \xrightarrow{x^p} R/x^{2p} R \xrightarrow{x^p} R/x^{3p} R \xrightarrow{x^p} \dots\},$$

and like before we have $\varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{p^{\ell}}R) \cong H_{\mathfrak{m}}^i(H_{(x)}^1(R))$. \square

We prove the main theorem of this article.

Theorem 3.6. *Let (R, \mathfrak{m}, k) be a local ring of prime characteristic $p > 0$ and $x \in \mathfrak{m}$ is a regular surjective element. If R/xR is F -injective, then R is also F -injective.*

Proof. Since R has a regular element x , we have $H_{\mathfrak{m}}^0(R) = 0$, so there is nothing to prove in the case $i = 0$. Consider the following commutative diagram of R -modules, where ρ_F denotes the natural Frobenius map

$$(3.4) \quad \begin{array}{ccccc} R/xR & \xrightarrow{\cdot x} & R/x^2R & \xrightarrow{\cdot x} & \dots \\ \downarrow \rho_F & & \downarrow \rho_F & & \\ F_*(R/x^pR) & \xrightarrow{\cdot x^p} & F_*(R/x^{2p}R) & \xrightarrow{\cdot x^p} & \dots \end{array}$$

Taking direct limits on the rows of Diagram 3.4 and applying $H_{\mathfrak{m}}^i(-)$, we get two directed systems $\{H_{\mathfrak{m}}^i(R/x^{\ell}R)\}_{\ell > 0}$ and $\{H_{\mathfrak{m}}^i(R/x^{p^{\ell}}R)\}_{\ell > 0}$ with Frobenius linear maps

$$\rho_{\ell, i}: H_{\mathfrak{m}}^i(R/x^{\ell}R) \rightarrow F_* H_{\mathfrak{m}}^i(R/x^{p^{\ell}}R)$$

which are injective for each $\ell > 0$ by Theorem 3.3. Thus the $\rho_{\ell, i}: H_{\mathfrak{m}}^i(R/x^{\ell}R) \rightarrow F_* H_{\mathfrak{m}}^i(R/x^{p^{\ell}}R)$ induce an injective Frobenius linear map

$$\rho_1 = \varinjlim_{\ell} \rho_{\ell, i}: \varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{\ell}R) \rightarrow F_* \varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{p^{\ell}}R),$$

since F_* commutes with \varinjlim by Lemma 3.4. The module $H_{(x)}^1(R)$ has a natural Frobenius action induced from the Frobenius on R and this induces a Frobenius linear action $\rho_2: H_{\mathfrak{m}}^i(H_{(x)}^1(R)) \rightarrow F_* H_{\mathfrak{m}}^i(H_{(x)}^1(R))$. Let ρ_3 denote the natural Frobenius action on $H_{\mathfrak{m}}^{i+1}(R)$.

It suffices to show that the following diagram commutes for each $i \geq 0$.

$$(3.5) \quad \begin{array}{ccccc} \varinjlim_{\ell} H_{\mathfrak{m}}^i(R/x^{\ell}R) & \xrightarrow{\alpha_1} & H_{\mathfrak{m}}^i(H_{(x)}^1(R)) & \xrightarrow{\beta_1} & H_{\mathfrak{m}}^{i+1}(R) \\ \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 \\ \varinjlim_{\ell} F_* H_{\mathfrak{m}}^i(R/x^{p^{\ell}}R) & \xrightarrow{F_* \alpha_2} & F_* H_{\mathfrak{m}}^i(H_{(x)}^1(R)) & \xrightarrow{F_* \beta_2} & F_* H_{\mathfrak{m}}^{i+1}(R) \end{array}$$

where α_1 and $F_*\alpha_2$ are the isomorphisms coming from Lemma 3.5, and β_1 and $F_*\beta_2$ are the isomorphisms coming from Lemma 2.2. Since ρ_1 is injective for $0 \leq i < \dim R - 1$, it follows that ρ_3 is injective for $0 \leq i < \dim R$ once we know the diagram commutes. We show the rest by splitting Diagram 3.5 into two commuting squares.

To show the first square in Diagram 3.5, note that this square is just applying $H_m^i(-)$ to the following square, where the vertical Frobenius linear maps are those induced by the natural Frobenius on R .

$$\begin{array}{ccc} \varinjlim_{\ell} R/x^{\ell}R & \xrightarrow{\cong} & H_{(x)}^1(R) \\ \downarrow & & \downarrow \\ \varinjlim_{\ell} F_*(R/x^{p^{\ell}}R) & \xrightarrow{\cong} & F_*H_{(x)}^1(R) \end{array}$$

The second square in Diagram 3.5 commutes, since $\mathbf{R}\Gamma_m \circ \mathbf{R}\Gamma_{(x)} \cong \mathbf{R}\Gamma_m$ in the derived category by [Wei94, Cor. 10.8.3] and we are simply applying each functor to the natural Frobenius $\rho_F: R \rightarrow F_*R$. That is to say, $\mathbf{R}\Gamma_m(\mathbf{R}\Gamma_{(x)}(\rho_F)) = \mathbf{R}\Gamma_m(\rho_F)$. \square

3.1. Deforming surjectivity of Frobenius linear maps. Clearly Frobenius linear maps are not generally surjective. Instead, it is essential to look at the span of the image under the Frobenius linear map. To make clear what this says, we start with a simple example.

Example 3.7. Let k be a perfect field of characteristic $p > 0$. The natural Frobenius linear action $k[x] \rightarrow F_*k[x]$ is R -linear and has image $F_*k[x^p]$ in $F_*k[x]$. So it is not surjective. However, it is surjective up to $F_*k[x]$ -span in the sense that F_*1 forms an $F_*k[x]$ -basis of $F_*k[x]$.

Say that an e -th iterated Frobenius linear map $\rho: M \rightarrow F_*^e N$ is *surjective up to $F_*^e R$ -span*, if the $F_*^e R$ -span of $\text{Im}(\rho)$ is equal to $F_*^e N$. This is just equivalent to having a set $\{a_i\}_{i \in \Lambda}$ of generators for M the $F_*^e R$ -submodule of $F_*^e N$ spanned by $\{\rho(a_i)\}_{i \in \Lambda}$ is equal to $F_*^e N$. This section investigates how this property deforms.

It is easy to see that if $\{\phi_i\}: \{M_i\} \rightarrow \{F_*M_i\}$ is a directed system of R -modules such that each $\phi_i: M_i \rightarrow F_*M_i$ is surjective up to F_*R -span, then so is $\phi = \varinjlim_i \phi_i$.

Lemma 3.8. *Let R be a commutative ring of characteristic $p > 0$ and assume that*

$$\begin{array}{ccccc} L & \xrightarrow{\alpha_1} & M & \xrightarrow{\alpha_2} & N \\ \rho_1 \downarrow & & \rho_2 \downarrow & & \rho_3 \downarrow \\ F_*L' & \xrightarrow{F_*\alpha'_1} & F_*M' & \xrightarrow{F_*\alpha'_2} & F_*N' \end{array}$$

*is a commutative diagram such that the top row is R -linear and exact and the bottom row is F_*R -linear and exact, and such that each ρ_i is a Frobenius linear map for $i = 1, 2, 3$. If ρ_1 and ρ_3 are surjective up to F_*R -span, then so is ρ_2 .*

Proof. Choose sets of generators of R -modules L , M , and N , say $\{x_i\}$, $\{y_j\}$, and $\{z_k\}$ respectively. Without loss of generality, we may assume $\{\alpha_1(x_i)\} \subseteq \{y_j\}$ and $\alpha_2(\{y_j\} \setminus \{\alpha_1(x_i)\}) = \{z_k\}$. It suffices to show each element of F_*M' can be presented as an F_*R -linear combination of $\{\rho_2(y_j)\}$. Pick $F_*m \in F_*M'$ and consider $F_*\alpha'_2(F_*m) \in F_*N'$. By hypothesis, we can write

$$(3.6) \quad F_*\alpha'_2(F_*m) = \sum_i F_*c_i \rho_3(z_i)$$

with $F_*c_i \in F_*R$. Now let $y'_i \in M$ be the inverse image of each $z_i \in N$ appearing in the equation (3.6). By our set up, we have $y'_i \in \{y_j\}$. By commutativity of the diagram, we also have $F_*m - \sum_i F_*c_i\rho_2(y'_i) \in \ker F_*\alpha'_2$. Since the bottom row is exact, one has $F_*m - \sum_i F_*c_i\rho_2(y'_i) = \sum_j F_*a_jF_*\alpha'_1(\rho_1(x_j))$ for some $F_*a_j \in F_*R$ and thus

$$F_*m = \sum_j F_*a_jF_*\alpha'_1(\rho_1(x_j)) + \sum_i F_*c_i\rho_2(y'_i) = \sum_j F_*a_j(F_*\alpha'_1(\rho_1(x_j))) + \sum_i F_*c_i\rho_2(y'_i),$$

which proves the lemma, since each $F_*\alpha'_1(\rho_1(x_j)) \in \{\rho_2(y_i)\}$. \square

As a corollary, we obtain the following.

Theorem 3.9. *Let (R, \mathfrak{m}) be a local ring of characteristic $p > 0$ with $x \in \mathfrak{m}$ a regular element. If the Frobenius action $H_{\mathfrak{m}}^i(R/xR) \rightarrow F_*H_{\mathfrak{m}}^i(R/xR)$ is surjective up to F_*R -span for all $i \geq 0$, the Frobenius action*

$$H_{\mathfrak{m}}^i(R) \rightarrow F_*H_{\mathfrak{m}}^i(R)$$

*is also surjective up to F_*R -span for all $i \geq 0$.*

Proof. Since x is a regular element, $H_{\mathfrak{m}}^0(R) = 0$ and there is nothing to prove in this case. Let $i \geq 0$. The e -th Frobenius linear action

$$H_{\mathfrak{m}}^i(R/xR) \rightarrow F_*^e H_{\mathfrak{m}}^i(R/xR),$$

induced by the e -th iterated Frobenius on R/xR , is surjective up to $F_*^e R$ -span by assumption. Recalling that the e -th iterated Frobenius $R/xR \rightarrow F_*^e(R/xR)$ factors as $R/xR \rightarrow F_*^e(R/x^{p^e}R) \rightarrow F_*^e(R/xR)$, $x \in \mathfrak{m}$ turns out to be a surjective element.

In this situation, Diagram 3.3 of Theorem 3.3 has exact rows. We proceed as in Theorem 3.3 by an induction on ℓ . Under the notation from Theorem 3.3, we just argued that $\rho_{1,i}$ is surjective up to F_*R -span. Working by induction for $\ell > 0$, we can assume $\rho_{\ell-1,i}$ is surjective up to F_*R -span and by Lemma 3.8, so is $\rho_{\ell,i}$. Thus $\rho_{\ell,i}$ is surjective up to F_*R -span for all $\ell > 0$.

Noting that $\rho_1 = \varinjlim_{\ell} \rho_{\ell,i}$, where ρ_1 is the same as given in Diagram 3.5 from Theorem 3.6, and noting that $\beta_2 \circ \alpha_1$ and $F_*\beta_2 \circ F_*\alpha_2$ are isomorphisms in Diagram 3.5, we see that β_3 is surjective up to F_*R -span as well. Putting this together we have shown the Frobenius action

$$H_{\mathfrak{m}}^{i+1}(R) \rightarrow F_*H_{\mathfrak{m}}^{i+1}(R)$$

is surjective up to F_*R -span for $i \geq 0$, as desired. \square

4. APPLICATIONS

Utilizing Theorem 3.6, we now describe two conditions for when F -injectivity deforms. One is a finite length condition on local cohomology modules, the other is F -purity.

4.1. Finite Length Cohomology. The first case that we can apply our main theorem to is one utilizing a finiteness condition on local cohomology modules.

Definition 4.1. For a local ring (R, \mathfrak{m}) , we say an R -module M has *finite local cohomology* (FLC) provided the local cohomology $H_{\mathfrak{m}}^i(M)$ has finite length as an R -module for all $i \leq \dim M - 1$.

Remark 4.2. In the literature when a local ring R has FLC it is called a *generalized Cohen-Macaulay ring*. When R has a dualizing complex, this means exactly that the non-CM locus is isolated [Sch75].

In the setting of a local ring (R, \mathfrak{m}) with $x \in \mathfrak{m}$ a regular element, we are most concerned with the R -modules R and $R/x^\ell R$; i.e., an infinitesimal neighborhood of the special fiber. We now show that FLC extends to such neighborhoods when imposed on the special fiber.

Lemma 4.3. *Let (R, \mathfrak{m}, k) be a local ring with $x \in R$ a regular element such that $\mathfrak{m}^s \cdot H_{\mathfrak{m}}^i(R/xR) = 0$ for some $s \geq 0$. For each $\ell > 0$, we have*

$$\mathfrak{m}^{s\ell} \cdot H_{\mathfrak{m}}^i(R/x^\ell R) = 0.$$

In particular, if R/xR has FLC, so does $R/x^\ell R$.

Proof. We show this by induction on ℓ . If $\ell = 1$, then this is just the hypothesis. Assume $\ell > 1$ and $\mathfrak{m}^{sj} \cdot H_{\mathfrak{m}}^i(R/x^j R) = 0$ for all $j < \ell$. The short exact sequence

$$0 \rightarrow R/x^{\ell-1}R \xrightarrow{x} R/x^\ell R \rightarrow R/xR \rightarrow 0,$$

induces a long exact sequence in local cohomology. We only need the portion

$$H_{\mathfrak{m}}^i(R/x^{\ell-1}R) \xrightarrow{\alpha} H_{\mathfrak{m}}^i(R/x^\ell R) \xrightarrow{\beta} H_{\mathfrak{m}}^i(R/xR),$$

which is an exact sequence of R -modules. Take an element $\eta \in H_{\mathfrak{m}}^i(R/x^\ell R)$ and $c \in \mathfrak{m}^s$. Then one has $\beta(c\eta) = c\beta(\eta) = 0$, which implies that $c\eta$ has a preimage $\theta \in H_{\mathfrak{m}}^i(R/x^{\ell-1}R)$ along α . By induction, we have $m \cdot \theta = 0$ for any $m \in \mathfrak{m}^{s(\ell-1)}$. Therefore, $\alpha(m \cdot \theta) = 0$ and $m \cdot c\eta = (mc) \cdot \eta = 0$. Since c and m were chosen arbitrarily, we have that $\mathfrak{m}^{s\ell} \cdot H_{\mathfrak{m}}^i(R/x^\ell R) = 0$. \square

Remark 4.4. We note that there was no restriction on characteristic on rings in Lemma 4.3.

An easy consequence of the FLC property is a result on surjective maps of local cohomology.

Lemma 4.5. *Let (R, \mathfrak{m}, k) be a local ring of characteristic $p > 0$ with perfect residue field k and $x \in \mathfrak{m}$ a regular element. Assume that R/xR is F -injective and FLC. For each $\ell > 0$, the surjection $R/x^\ell R \rightarrow R/xR$ induces a surjection*

$$H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/xR)$$

for each $i \geq 0$.

Proof. By Lemma 2.5, since R/xR has FLC and is F -injective with perfect residue field, the e -th iterated Frobenius action

$$H_{\mathfrak{m}}^i(R/xR) \rightarrow F_*^e H_{\mathfrak{m}}^i(R/xR)$$

induced by Frobenius on R/xR is surjective. For $\ell > 0$, choose $e \gg 0$ so that the surjection $R/x^{p^e}R \rightarrow R/xR$ factors as $R/x^{p^e}R \rightarrow R/x^\ell R \rightarrow R/xR$. This induces a composition of maps:

$$H_{\mathfrak{m}}^i(R/xR) \rightarrow F_*^e H_{\mathfrak{m}}^i(R/x^{p^e}R) \rightarrow F_*^e H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow F_*^e H_{\mathfrak{m}}^i(R/xR).$$

The composition is surjective and so $H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/xR)$ must be. \square

Corollary 4.6. *Let (R, \mathfrak{m}, k) be a local ring of characteristic p with perfect residue field and $x \in \mathfrak{m}$ a regular element. If R/xR has FLC and is F -injective, then R is F -injective.*

Proof. By Lemma 4.5 the hypothesis of Theorem 3.6 are satisfied. \square

Immediately this shows that potential counterexamples to the deformation of F -injectivity in nice geometric settings must have dimension at least 4.

Corollary 4.7. *If (R, \mathfrak{m}, k) is a complete local ring of characteristic $p > 0$ with perfect residue field and dimension at most 4 and $x \in \mathfrak{m}$ is a regular element with R/xR normal and F -injective, then R is F -injective.*

Proof. Since R/xR is a local normal domain and $x \in \mathfrak{m}$ is a regular element, R is also normal by ([Gro65] 5.12.7). In particular, R is a domain and equidimensional. Since $\dim R \leq 4$, one has $\dim R/xR \leq 3$. By normality of R/xR , it satisfies Serre's condition S_2 , therefore the non-CM locus is isolated, hence R/xR has FLC and by Corollary 4.6, R must be F -injective. \square

Example 4.8. We give an example of a local ring (R, \mathfrak{m}) with $x \in \mathfrak{m}$ a regular element, such that R/xR has FLC and is F -injective and R does not have FLC. Let

$$A = \mathbb{F}_p[[a, b, c, d]]/(a, b) \cap (c, d).$$

Then A has FLC (even Buchsbaum; see [GO83]), non Cohen-Macaulay, and F -pure by Fedder's criterion [Fed83]. Then $R := A[[x]]$ is clearly F -injective, but the non-CM locus of R is defined by the non-maximal ideal $\mathfrak{n}R$ for the maximal ideal \mathfrak{n} of A . Thus R does not have FLC.

Remark 4.9. The assumption that the residue field of R is perfect is necessary in the proof of the main theorem. If R is F -injective and contains a non-perfect field K , it is not necessarily true that $R \otimes_K K^{1/p}$ is F -injective. For example, put $K := \mathbb{F}_p(x)$. Then $R := K[t]/(t^p - x)$ is a field. However, $R \otimes_K K^{1/p} \cong K^{1/p}[t]/(t - x^{1/p})^p$ is not reduced. But F -injective local rings need to be reduced.

4.2. F -splitting and F -injectivity. The second application concerns F -purity. We utilize work of L. Ma [Ma] building on work by Enescu and Hochster [EH]. The language used in [EH] is in terms of $R\{F\}$ -modules which are modules over a ring R with a specified Frobenius action. For such a module M with a distinguished Frobenius action $\rho: M \rightarrow F_*M$, a submodule $N \subset M$ is called F -compatible, provided that $\rho(N) \subseteq F_*N$. Ma showed that F -split local rings have local cohomology modules, which when equipped with the natural Frobenius action, satisfy an interesting condition, originally introduced in [EH].

Definition 4.10. ([EH, Def. 4.6]) Let (R, \mathfrak{m}) be a local ring. An R -module M with a Frobenius action ρ is called *anti-nilpotent*, provided for any submodule F -compatible submodule N (i.e., $\rho(N) \subseteq F_*N$), the induced action of ρ on M/N is injective.

Theorem 4.11. Let (R, \mathfrak{m}, k) be a local ring of characteristic $p > 0$ with perfect residue field and $x \in \mathfrak{m}$ a regular element. If R/xR is F -split then R is F -injective.

Proof. From Theorem 3.6, one need only show that $H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/xR)$, which is induced by the surjection $R/x^\ell R \rightarrow R/xR$, is surjective. Denote its cokernel by C . It suffices to show that $C = 0$. Consider the exact sequence $H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow H_{\mathfrak{m}}^i(R/xR) \rightarrow C \rightarrow 0$. Denote by $\rho_{\ell, i}^e: H_{\mathfrak{m}}^i(R/x^\ell R) \rightarrow F_*^e H_{\mathfrak{m}}^i(R/x^\ell R)$ induced naturally by the Frobenius on R composed with the natural surjection.

The map $\rho_{1, i}^e$ induces a Frobenius linear map $C \rightarrow F_*^e C$ and denote this by ρ_C^e . These Frobenius linear maps fit together to give a commutative diagram with exact rows since F_*^e is exact for all e .

$$\begin{array}{ccccccc} H_{\mathfrak{m}}^i(R/x^\ell R) & \longrightarrow & H_{\mathfrak{m}}^i(R/xR) & \longrightarrow & C & \longrightarrow & 0 \\ \rho_{\ell, i}^e \downarrow & & \rho_{1, i}^e \downarrow & & \rho_C^e \downarrow & & \downarrow \\ F_*^e H_{\mathfrak{m}}^i(R/x^\ell R) & \longrightarrow & F_*^e H_{\mathfrak{m}}^i(R/xR) & \longrightarrow & F_*^e C & \longrightarrow & 0 \end{array}$$

The image of $H_{\mathfrak{m}}^i(R/x^\ell R)$ in $H_{\mathfrak{m}}^i(R/xR)$ is certainly F -compatible. Since we assume R/xR is F -split, the module $H_{\mathfrak{m}}^i(R/xR)$ is anti-nilpotent by [Ma, Thm. 3.7] and so the Frobenius action ρ_C^e

on C is injective. Note also that when $e \gg 0$, the map $\rho_{1,i}^e$ factors as $H_{\mathfrak{m}}^i(R/xR) \rightarrow F_*^e(R/x^{p^e}R) \rightarrow F_*^e(R/x^\ell R) \rightarrow F_*^e(R/xR)$. So we may define the map φ to make the following diagram commute:

$$(4.1) \quad \begin{array}{ccccccc} H_{\mathfrak{m}}^i(R/x^\ell R) & \longrightarrow & H_{\mathfrak{m}}^i(R/xR) & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow \rho_{\ell,i}^e & \nearrow \varphi & \downarrow \rho_{1,i}^e & & \downarrow \rho_C^e & & \downarrow \\ F_*^e H_{\mathfrak{m}}^i(R/x^\ell R) & \longrightarrow & F_*^e H_{\mathfrak{m}}^i(R/xR) & \longrightarrow & F_*^e C & \longrightarrow & 0 \end{array}$$

We show that $C = 0$ by utilizing the diagram chase on (4.1). Let $z \in C$. As such, it has a preimage $z' \in H_{\mathfrak{m}}^i(R/xR)$. By commutativity of the diagram, it follows that $\rho_{1,i}^e(z')$ has preimage $z'' = \varphi(z')$. As the bottom row is exact, z'' maps to $\rho_C^e(z)$ which is zero. However ρ_C^e was shown to be injective, this implies that $z = 0$ and therefore $C = 0$, as desired. \square

Remark 4.12. Under an F -finite assumption, Theorem 4.11 says that F -purity deforms to F -injectivity. Enescu obtained some results on finiteness property on local cohomology modules of finite length [Ene12, Thm. 7.14].

Example 4.13. Let (R, \mathfrak{m}) be an F -finite local ring. If R/xR is F -pure for a regular element $x \in \mathfrak{m}$, but R is not F -pure, then R is an example which is F -injective, but not F -pure. Such examples abound. For example, in [Sin99a] it is shown that $R := \mathbf{F}_p[[X, Y, Z, W]]/(XY, XW, W(Y - Z^2))$ is not F -pure, but $R/zR = \mathbf{F}_p[[X, Y, W]]/(XY, XW, WY)$ is. This also means that our main result serves as a way for checking F -injectivity by taking specialization, *i.e.*, checking that R/zR is F -pure.

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APPENDIX A. F -INJECTIVITY AND DEPTH

by Karl Schwede and Anurag K. Singh

Our goal here is to prove a prime characteristic analog of a result of Kollár and Kovács, [KK10, Theorem 7.12]: if $X \rightarrow B$ is a flat family with Du Bois fibers, such that the generic fiber is Cohen-Macaulay (respectively S_k), then all fibers of the map $X \rightarrow B$ are Cohen-Macaulay (respectively S_k). The prime characteristic version of this is Theorem A.3 below. As applications of this theorem, we extend a result of Fedder and Watanabe [FW89, Proposition 2.13] to the case where R is not a priori assumed to be Cohen-Macaulay, see Corollary A.4, and also obtain a new result on the deformation of F -injectivity, Corollary A.5.

We begin with some preliminary observations:

Lemma A.1. *Let (R, \mathfrak{m}) be a local ring; set d to be the depth of R . Suppose there exists a regular element f in R such that the Frobenius action on $H_{\mathfrak{m}}^{d-1}(R/fR)$ is injective. Then the map*

$$H_{\mathfrak{m}}^d(R) \xrightarrow{f^{p-1}F} H_{\mathfrak{m}}^d(R);$$

is injective; in particular, the Frobenius action on $H_{\mathfrak{m}}^d(R)$ is injective.

Proof. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & R/fR \longrightarrow 0 \\ & & \downarrow f^{p-1}F & & \downarrow F & & \downarrow F \\ 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & R/fR \longrightarrow 0. \end{array}$$

Since R/fR has depth $d-1$, applying the functor $H_{\mathfrak{m}}^{\bullet}(\)$ yields the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathfrak{m}}^{d-1}(R/fR) & \longrightarrow & H_{\mathfrak{m}}^d(R) & \xrightarrow{f} & H_{\mathfrak{m}}^d(R) \longrightarrow H_{\mathfrak{m}}^d(R/fR) \\ & & \downarrow F & & \downarrow f^{p-1}F & & \downarrow F \\ 0 & \longrightarrow & H_{\mathfrak{m}}^{d-1}(R/fR) & \longrightarrow & H_{\mathfrak{m}}^d(R) & \xrightarrow{f} & H_{\mathfrak{m}}^d(R) \longrightarrow H_{\mathfrak{m}}^d(R/fR). \end{array}$$

The map $f^{p-1}F$ is injective if and only if it is injective when restricted to the socle of $H_{\mathfrak{m}}^d(R)$. The socle is annihilated by f , and thus lies in the image of $H_{\mathfrak{m}}^{d-1}(R/fR)$. But the Frobenius action on $H_{\mathfrak{m}}^{d-1}(R/fR)$ is injective by assumption. \square

The next lemma is the main ingredient in the proof of Theorem A.3. For a local ring (R, \mathfrak{m}) , we use $\text{Spec}^{\circ} R$ to denote the *punctured spectrum* of R , i.e., the set $\text{Spec } R \setminus \{\mathfrak{m}\}$. The F -finite hypothesis in the sequel ensures the existence of a dualizing complex by Gabber, [Ga04, Remark 13.6].

Lemma A.2. *Let (R, \mathfrak{m}) be an F -finite local ring. Suppose there exists a regular element f in R such that R/fR is F -injective.*

If $R_{\mathfrak{p}}$ satisfies the Serre condition S_k for each \mathfrak{p} in $\text{Spec}^\circ R$, then R satisfies S_k .

Proof. Let d be the depth of R . If R does not satisfy S_k , then $d < k$.

The module $H_{\mathfrak{m}}^d(R)$ is nonzero, but has finite length since $R_{\mathfrak{p}}$ satisfies S_k for each prime ideal \mathfrak{p} in $\text{Spec}^\circ R$. We claim that $\mathfrak{m}H_{\mathfrak{m}}^d(R) = 0$. Because it has finite length, the module $H_{\mathfrak{m}}^d(R)$ is annihilated by \mathfrak{m}^q for some $q = p^e$. For each $x \in \mathfrak{m}$ and $\eta \in H_{\mathfrak{m}}^d(R)$, it follows that $x^q F^e(\eta) = 0$. But the Frobenius action on $H_{\mathfrak{m}}^d(R)$ is injective by Lemma A.1, so $x\eta = 0$, which proves the claim.

But then $f^{p-1}H_{\mathfrak{m}}^d(R) = 0$. Since $f^{p-1}F: H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(R)$ is injective by Lemma A.1, we must have $H_{\mathfrak{m}}^d(R) = 0$, which is a contradiction. \square

Theorem A.3. *Let R be an F -finite local ring. Suppose there exists a regular element f in R such that R/fR is F -injective.*

If the localization $R_f = R[f^{-1}]$ satisfies the Serre condition S_k for a positive integer k , then R satisfies condition S_k . In particular, if R_f is Cohen-Macaulay, then R is Cohen-Macaulay.

Proof. If not, take a prime \mathfrak{q} that is minimal with respect to the property that $R_{\mathfrak{q}}$ does not satisfy S_k . As R_f is S_k by assumption, it follows that $f \in \mathfrak{q}$. Since it is a localization of an F -injective ring, the ring $(R/fR)_{\mathfrak{q}} = R_{\mathfrak{q}}/fR_{\mathfrak{q}}$ is F -injective, see, for example, [Sch09, Proposition 4.3]. But $(R_{\mathfrak{q}})_{\mathfrak{p}}$ satisfies condition S_k for each prime ideal \mathfrak{p} in $\text{Spec}^\circ R_{\mathfrak{q}}$, so $R_{\mathfrak{q}}$ satisfies S_k by Lemma A.2. This is a contradiction. \square

The following corollary was proved as [FW89, Proposition 2.13] under the additional hypothesis that R is Cohen-Macaulay:

Corollary A.4. *Let R be an F -finite local ring. Suppose there exists a regular element f in R such that R/fR is F -injective. If R_f is F -rational, then R is F -rational.*

Proof. Theorem A.3 implies that R is Cohen-Macaulay. But then R is F -rational by [FW89, Proposition 2.13]; Fedder and Watanabe require R_f to be regular in the statement of the proposition, but their proof works verbatim if some power of f is a parameter test element, and this is indeed the case by [Ve95, Theorem 1.13]. \square

Fedder [Fed83, Theorem 3.4 (1)] proved that F -injectivity deforms in the case of Cohen-Macaulay rings; we extend this as follows:

Corollary A.5. *Let R be an F -finite local ring. If $f \in R$ is a regular element such that R/fR is F -injective, and R_f is Cohen-Macaulay, then R is F -injective.*

Proof. Theorem A.3 implies that the ring R is Cohen-Macaulay; we may then use [Fed83, Theorem 3.4.1]. \square

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